Combinatorial Optimization : Matchings, Matroids and the Travelling Salesman

On the Crossroad of the Salesman and the Postman

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Abstract

The purpose of this handout is to provide "warming up" exercises for my course, which will be alternating between lecture, exercise solving, and "group discovery" style. It presupposes an introductory course in graph theory and combinatorial optimization, so the most well-known definitions are omitted. They can be found in any introductory course or book. The last part of the course will show an advanced application of the learnt methods and acquired skills.

The intent of the course is to provide sufficient foundations in combinatorial optimization so that you could then progress alone and orient yourself correctly, and try to solve your optimization problems with appropriate methods. The exercises presented here prepare the communication of these methods.

Some of these exercises will be restated and used during the course, and the full solution will be given whenever it is necessary. Some others intend to revise some material of introductory courses, in a way helpful for the course.

Have fun with them !

1 Help

Key-words : flows, matchings, matroids, packing and covering, linear programming, polyhedral combinatorics, minmax theorems of combinatorial optimization, algorithmic proofs, TSP (travelling salesman problem), T-joins, bin packing, complexity theory, connectivity, ear-theorems, matroid intersection, P, NP, RP, coNP, NP-complete, approximation algorithms, APX-hard.

Advised Preliminary Knowledge : We will define the notions we are dealing with and briefly repeat the theorems and algorithms of introductory courses, but it helps if you are already familiar with the following :

Shortest paths in undirected graphs and digraphs, network flows (Ford and Fulkerson's algorithm, max flow min cut theorem, min cost flows), bipartite matchings,

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Introduction to complexity theory (P, NP, coNP, PTAS, APX-complete, approximation ratio ...) Kruskal's algorithm for minimum weight spanning trees, definitions and basic facts about matroids, linear programming (simplex method, duality theorem), similar basic knowledge of first courses of graph theory or operations research.

We will not use any knowledge about non-bipartite matchings or generalizations, a simple approach to these will be fully included. (Some exercises below lead to Edmonds' algorithm and Tutte's theorem.)

For patches filling knowledge holes, or familiarizing in advance some new knowledge of the course, I advise you have a look at one or several of the following textbooks. In case of additional interest, you can deepen your knowledge with more advanced books or research articles.

Textbooks :

Korte, Vygen : Combinatorial Optimization, New Edition, (Springer 2012).

Lovász : Combinatorial Problems and Ecercises (Akadémiai Kiadó)

Lovász, Plummer : Matching Theory (Akadémiai Kiadó)

Schrijver : Combinatorial Optimization (Springer)

Lau, Ravi, Singh : Iterative Methods in Combinatorial Optimization (2011)

Shmoys, Williamson : The Design of Approximation algorithms (2011)

More advanced books and articles related to the last part of the course :

Frank : Connexions in Combinatorial Optimization

Sebő, Vygen (2012) : Shorter Tours by Nicer Ears : 7/5-approximation for graphic TSP, 3/2 for the path version, and 4/3 for two-edge-connected subgraphs,
http ://arxiv.org/abs/1201.1870

Sebő (2012) Eight-Fifth Approximation for TSP Paths,
http ://arxiv.org/abs/1209.3523

Sebő (1990) Undirected distances and the postman-structure of graphs, Journal of Combinatorial Theory, Series B, vol. 49,no. 1,1990.

2 Instructions for the exercises

We will discuss solutions of (the following or other) exercises whenever they are needed during the course.

Even if there is not enough time to solve them all, having a look and little thinking about them in advance may make it easier to discover a solution. Most of the exercises I state contain an idea that may be useful to know for the course.

Let G be an undirected graph in this document directed graphs will not occur ; $V(G)$ denotes the set of its vertices, $E(G)$ is its edge-set ; $\nu(G)$ denotes the matching number of G, that is, the maximum size of a set of disjoint edges in G ; $\tau(G)$ is the minimum vertex cover of G, that is, a set of vertices of minimum cardinality that meet every edge of G.

If you don't understand an exercise, or you have any question or comment concerning the course or this hand-out, please, don't hesitate contacting me by e-mail ; I will try to answer if I am online.

If you cannot solve an exercise, don't worry : to have understood it, to have thought about it and to have realized the difficulty will already be helpful enough for the course.

3 Matchings

Exercise 1 Let G be a bipartite graph, and $uv \in E(G)$. Then either $\nu(G-u) < \nu(G)$, or $\nu(G-v) < \nu(G)$. Deduce from this a simple inductive proof of König's theorem $\nu(G) = \tau(G)$ for every bipartite graph G.

Exercise 2 Let G be a graph, and $uv \in E(G)$. Then either $\nu(G-u) < \nu(G)$, or $\nu(G-v) < \nu(G)$, or else for any maximum matching M_u of $G-u$, and M_v of $G-v$: $M_u \cup M_v$ contains an (u,v)-path P alternating between M_u and M_v.

If G is a graph, and $X \subseteq E(G)$, then G/X denotes the graph we get from G by identifying the endpoints of the edges in X (and deleting the edges induced by X).

Exercise 3 Let G be a graph, and M a maximum matching in G, moreover $uv \in E(G)$, $\nu(G-u) = \nu(G)$, $\nu(G-v) = \nu(G)$. Then for the alternating path P of the previous exercise, the minimum number of uncovered vertices in G/P is the same as in G.

Exercise 4 Deduce by induction, using exercises 2 and 3 the theorem of Berge-Tutte : the minimum number of vertices not covered by a matching is equal to the maximum of $q(X) - |X|$, where $q(X)$ denotes the number of odd components of $G - X$. If you know Edmonds' algorithm deduce also a proof of its correctness.

If for a set X the value of $q(X) - |X|$ is maximum, then it is called a Tutte-set.

Exercise 5 If $v \in V(G)$ is contained in some Tutte-set then it is covered by every maximum matching of G

4 Postman tours

A *postman tour* is a closed walk which uses every edge of the graph at least once. Let us call a set *postman set* if its deletion leads to a graph with all degrees even (but not necessarily connected).

Exercise 1 The minimum length of a postman tour is equal to $|E(G)| + \tau$ where τ is the minimum cardinality of a postman set.

Exercise 2 A postman set P has minimum cardinality if and only if there is no circuit of negative weight according to the weight function which is -1 on the edges of P and 1 on the other edges.

Exercise 3 What is the mistake in the following algorithm (it contains the maximization of matchings, why ?) : use a shortest path algorithm to find a negative circuit ; if there is one, interchange the -1 and $+1$ weights, improving the postman set ; if there is none, we are done by Exercise 2.

5 Conservative weightings

A weighting of the edges of a graph is called *conservative* if there is no circuit of negative weight. The *distance* between pairs of points is the minimum weight of paths.

Exercise 1 In a graph given with a conservative weighting, changing the sign of the edges on a 0-weight circuit the distances do not change.

Hint : Take the symmetric difference of a shortest (a, b)-path and the 0 weight circuit and observe that what you get has odd degree in a and b and even degree everywhere else.

Exercise 2 Given a graph with two ± 1 conservative weightings, but where the parities of the number of negative edges adjacent to the vertices are the same, the distances between any two vertices are also the same in the two graphs.

Exercise 3 In a graph G given with a conservative weighting and $a \in V(G)$, a vertex $b \neq a$ whose distance from a is minimum is adjacent to exactly one negative edge.

Exercise 4 In a ± 1-weighted conservative bipartite graph, contracting the edges adjacent to a vertex b whose distance is minimum from a vertex a, the obtained graph is also conservative with the original weighting, and the distances of the vertices of G from a do not change.

Exercise 5 In a ± 1-weighted conservative bipartite graph there exist edge-disjoint cuts covering all the negative edges so that each cut contains exactly one negative edge.

6 T-joins

Let $T \subseteq V(G)$, $|T|$ even. A *T-join* is a set of edges whose set of odd degree vertices is exactly T. Note that a postman set is a T_G-join, where T_G is the set of odd degree vertices of G. Let $\tau(G, T)$ be the minimum cardinality of a postman set in G, $\tau := \tau(G, T_G)$.

Exercise 1 In a bipartite graph τ is equal to the maximum number of pairwise edge-disjoint cuts defined by bipartitions $\{X, Y\}$ of $V(G)$, where X contains an odd number of vertices of odd degree. Is this a 'good characterization' (a theorem that puts the corresponding decision problem in NP intersection coNP) ? Can you say something about non-bipartite graphs ?

Hint : Use Exercise 5.5.

Exercise 2 Can you generalize Exercise 3.2 and the preceding Exercise 6.1 from T_G-joins (which are exactly the postman sets) to arbitrary T-joins ?

Exercise 3 Does Exercise 6.1 have a weighted generalization ?

7 Matroid operations

If you never heard about matroids skip this series. We will define matroids at the course, and if necessary, spend time with the main examples, some equivalent axioms and other basics. If you try though, you may be rewarded by Exercise 6.

A *minor* of the matroid $M = (S, \mathcal{F})$ is a matroid obtained from M by a succession of *deletions* and *contractions* of elements, that is :
$M \setminus e := M - e := (S \setminus e, \mathcal{F} \setminus e)$, $\qquad M/e := (S \setminus e, \mathcal{F}/e)$,
where $\mathcal{F} \setminus e = \{F \in \mathcal{F} : F \subseteq S \setminus e\}$, $\qquad \mathcal{F}/\{e\} = \{F \in \mathcal{F} : F \subseteq S \setminus e, F \cup \{e\} \in \mathcal{F}\}$.
The *dual* $M^* = (S, \mathcal{B}^*)$ of $M = (S, \mathcal{B})$ (matroids defined with the basis axioms) is defined as $\mathcal{B}^* := \{S \setminus B : B \in \mathcal{B}\}$. **!**

The *sum* of two matroids $M = (S, \mathcal{F}_1)$, $M = (S, \mathcal{F}_2)$: $M = (S, \mathcal{F})$, where $\mathcal{F} := \{F = F_1 \cup F_2 : F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2\}$.

Exercise 1 Show that the result of all these operations is a matroid.

Exercise 2 Show $(M \setminus e)/f = (M \setminus f)/e$, that is, the result of a succession of deletions and contractions does not depend on the order of these operations.

Exercise 3 Show $(M \setminus e)^* = M^*/e$.

Exercise 4 Show that the rank function of the dual of a matroid with rank function r is : $r^*(X) = |X| - (r(S) - r(S \setminus X))$.

Exercise 5 Show that in the special case of graphic matroids these operations specialize to the well-known graph operations of the same name. In particular, if G is a planar graph, $M^*(G) = M(G^*)^*$; in addition, the circuits of G are the cuts of G^*.

Exercise 6 Prove Euler's formula : suppose $G = (V, E)$ is a connected planar graph, with f faces. Then $|V| - |E| + f = 2$.

Hint : Use the preceding exercise to show that deleting a spanning tree from G you get a spanning tree of G^*, and therefore $(|V| - 1) + (f - 1) = |E|$.